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RESEARCH ARTICLE

Distribution-free control chart for Bivariate Process

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Abstract

Nonparametric or distribution-free control chart is useful in statistical process control when the underlying process distribution is unknown or is not likely to be normal. In this study, a distribution-free control chart based on Hodges sign test is developed for detecting the shifts in the location of a bivariate process. A simulation study is conducted to study the average run length performance of the proposed chart and is compared with the parametric control chart under normal and heavy tailed distributions. The study indicates that the proposed control chart is efficient in detecting various magnitudes of shifts when underlying process distribution is heavy tailed.

Keywords: Nonparametric, average run length, distribution-free, Hodges sign test, bivariate process.

Introduction

Control charts are useful tools for monitoring a manufacturing process. Univariate control charts are used to monitor processes that manufacture products with a single quality characteristic of interest. There are many situations in which a process is characterized by more than one correlated quality characteristics. Multivariate control charts are needed to monitor such processes. Most of the standard multivariate control charts that have been developed in literature are based on the assumption that the process data follow a multivariate normal distribution. The statistical properties of commonly employed control charts are exact only if the assumption is satisfied; however, underlying process is not normal in many practical situations and as a result the statistical properties of standard charts can highly affected in such situations. Distribution-free or nonparametric control charts are parallel alternatives if one is concerned about non-normality and contamination.

A formal definition of nonparametric or distribution-free control chart is given in terms of its in-control run length distribution. If the in-control run length distribution is same for every continuous distribution then the chart is called distribution-free. Several nonparametric control charts are proposed for monitoring location of a univariate process. Some of these are based on signs and/or rank statistics by assuming a known in-control target value for process location. Chakraborti *et al.* (2001) presented an extensive literature on univariate nonparametric control charts. For monitoring multivariate process, a very few nonparametric control charts are available in literature. Hayter and Tusi (1994) proposed a Shewhart-type multivariate nonparametric control scheme for the mean vector. Kapatou and Reynolds (1994, 1998) proposed EWMA-type multivariate control charts for groups based on the sign and signed-rank statistics.

Liu (1995) proposed a Shewhart-type multivariate nonparametric control chart based on simplicial data depth. Qiu and Hawkins (2003) suggest a nonparametric multivariate CUSUM procedure based on the antiranks of the measurement component. Das (2009) proposed a multivariate nonparametric control chart based on bivariate sign test. Since few works are reported in the literature on multivariate nonparametric control charts for monitoring location of the multivariate process, the purpose of this study is to develop Shewhart-type nonparametric control chart for monitoring a process location of a bivariate symmetric process. It is assumed that the in-control (target) values and certain correlations for the variables being monitored are pre-specified. The control chart statistic used is the Hodges bivariate sign test statistic. The proposed distribution-free control chart is compared with the corresponding parametric chart for the normal and the double exponential distributions. Their efficiencies are compared using the average run length (ARL) criterion computed with simulation.

Materials and methods

Hotelling's T^2 control chart

Let $\boldsymbol{X}_i = (X_{1i}, X_{2i}), i = 1,2,...,n$ be a subgroup sample from a symmetric bivariate distribution with the location $\boldsymbol{\mu}$ and the covariance matrix Σ. Let $\boldsymbol{\mu}_0$ and Σ_0 be the desired process center and the covariance matrix respectively. Without loss of generality, we assume that $\boldsymbol{\mu}_0 = (0,\ 0)'$ and $\Sigma_0 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $-1 < \rho < 1$.

We are interested in detecting shifts in $\boldsymbol{\mu}$. When the distribution of observations is bivariate normal, Hotelling's T^2 is an appropriate chart for this problem with charting statistic

$$T^2 = n\ (\overline{\boldsymbol{X}} - \boldsymbol{\mu}_0)'\ \Sigma_0^{-1}\ (\overline{\boldsymbol{X}} - \boldsymbol{\mu}_0), \qquad (1)$$

Where $\overline{X}$ is the sample mean vector, μ_0 is the in-control mean vector and Σ_0 is the covariance matrix. When the process is in-control, $\mu = \mu_0$ the statistic T^2 is distributed as chi-square variate with 2 degrees of freedom. If $\mu \neq \mu_0$, the statistic T^2 is distributed as a non-central chi-square variable with 2 degrees of freedom and a non-centrality parameter λ^2.

The value of non-centrality parameter is

$$\lambda^2 = n(\mu - \mu_0)^{'} \Sigma_0^{-1} (\mu - \mu_0) = n\, d^2 ,$$

Where $d = \sqrt{(\mu - \mu_0)^{'} \Sigma_0^{-1} (\mu - \mu_0)}$ (2)

is the Mahalanobis distance, used to measure a change in the process mean vector.

The ARL values of this chart can be calculated as

$$\text{ARL} = \frac{1}{\text{P}}, \qquad (3)$$

Where $\text{P} = \text{P}(T^2 > UCL \mid d)$.

The on-target value of P is determined as

$$\text{P}(0) = \text{P}(T^2 > UCL \mid d = 0) = 1 - \text{F}_\text{p}(UCL), \qquad (4)$$

Where, $\text{F}_\text{p}(.)$ is the cumulative distribution function of a central chi-square distribution with *p* degrees of freedom.

Nonparametric control chart based on Hodges Bivariate sign test

For a bivariate subgroup sample $(X_{1i}, X_{2i}), i = 1,2,\ldots, n$, denote the direction angle of (X_{1i}, X_{2i}) by θ_i. Then $\theta_i^* = \theta_i + \pi \pmod{2\pi}$ is the direction angle of $(-X_{1i}, -\text{X}_{2i})$.

Write $\theta_1^{'} < \theta_2^{'} < \ldots < \theta_{2n}^{'}$ ordered angles in the set $\{\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*\}$.

$$\text{Define } z_i = 1, \quad \theta_i^{'} \in \{\theta_1, \ldots, \theta_n\} \qquad (5)$$
$$= 0, \quad \theta_i^{'} \in \{\theta_1^*, \ldots, \theta_n^*\}, \quad i = 1, \ldots, n$$

Based on sign vector z, Oja and Nyblom (1989) have given several bivariate generalizations of the sign test for testing the location parameter of a univariate distribution. These tests include the bivariate sign tests proposed by Hodges (1955). They presented Hodges bivariate sign test statistic in the following form.

$$\text{H} = \max_{0 \leq k \leq n-1} \left| \sum_{i=1}^{n} z_{k+i} - \frac{n}{2} \right| \qquad (6)$$

We use sign test statistic H as control chart statistics for our proposed nonparametric control chart. The chart is referred as NP-H chart.

Steps for the proposed non-parametric control chart

1. Take a bivariate sample (X_{1i}, X_{2i}) of size *n* at each inspection point.
2. Calculate control statistic H.
3. Choose an upper control limit.
4. Plot H in the chart.
5. If any point goes beyond the limit, the process is considered to be *out-of-control* and it is as an indication that there is a shift in process location.

Results and discussion

Performance comparison

To examine the ability of proposed NP-H chart to detect location shift in bivariate process, we consider underlying process distributions as bivariate normal and bivariate double exponential with on-target mean vector $\mu_0 = (0,\ 0)^{'}$, $\Sigma_0 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and the sample size *n* = 25.

The bivariate double exponential distribution is chosen to study, how the heavy tailed distribution would affect the performance of the control chart. In vector notation we denote the shift in μ_0 by $\delta = (\delta_1, \delta_2)^{'}$. The statistical distance of μ from μ_0 is $d = \sqrt{\delta^{'} \Sigma_0^{-1} \delta}$. For a given value of *d*, there are many combinations of shifts in the two means that will produce this *d*, but only two types of shifts are simulated. Shift of the type $\delta = (\delta, 0)^{'}$ represent one variable shift means a shift in one variable only, that is, $\delta_1 = \delta,\ \delta_2 = 0$, shift of the form $\delta = (\delta, \delta)^{'}$ represent an equal shift, means that both means have shifted so that $\delta_1 = \delta_2 = \delta$. Shifts of these types are investigated for the distances of 0.2, 0.4, 0.6, 0.8 and 1.0 from in-control location vector $\mu_0 = (0,0)^{'}$.

The upper control limits of the charts are then adjusted so that all charts have approximately the same in-control ARL value 380. Except for the Hotelling's T^2 chart under bivariate normal distribution, the ARL values of the various control charts are computed using 10000 simulations when underlying process distributions are bivariate normal and bivariate double exponential.

Table 1 and Table 2 presents the ARL values of the proposed nonparametric control chart and Hotelling's T^2 chart when underlying process distributions are bivariate normal and bivariate double exponential with sample size *n* = 25 and $\rho = 0$ and 0.6 respectively.

Table 1. ARL comparison with ρ = 0.0.

Shift d	Shift type	Bivariate normal: Hotelling's T^2 UCL = 11.88	Bivariate normal: NP-H UCL = 8.5	Bivariate double exponential: Hotelling's T^2 UCL = 12.68	Bivariate double exponential: NP-H UCL = 8.5
0.0	None	380	380	380	380
0.2	One var.	68.67	113.69	82.35	55.14
	Equal		112.51	82.42	58.08
0.4	One var.	9.53	21.35	11.80	8.54
	Equal		21.24	11.79	9.09
0.6	One var.	2.59	5.90	2.95	2.91
	Equal		5.84	2.91	2.99
0.8	One var.	1.33	2.43	1.40	1.65
	Equal		2.52	1.39	1.58
1.0	One var.	1.05	1.49	1.06	1.23
	Equal		1.46	1.06	1.18

Table 2. ARL comparison with ρ = 0.6.

Shift d	Shift type	Bivariate normal: Hotelling's T^2 UCL = 11.88	Bivariate normal: NP-H UCL = 8.5	Bivariate double exponential: Hotelling's T^2 UCL = 16.58	Bivariate double exponential: NP-H UCL = 8.5
0.0	None	380	380	380	380
0.2	One var.	68.67	113.79	98.07	77.08
	Equal		113.94	173.22	37.98
0.4	One var.	9.53	21.42	17.46	13.14
	Equal		21.26	20.23	5.77
0.6	One var.	2.59	5.97	4.55	4.26
	Equal		5.85	5.38	2.13
0.8	One var.	1.33	2.48	1.94	2.16
	Equal		2.43	1.82	1.33
1.0	One var.	1.05	1.47	1.24	1.46
	Equal		1.46	1.13	1.10

Examination of Table 1 and 2 leads to the following findings:

- For monitoring a process operating under a bivariate normal distribution, we observe that, the proposed control chart is less efficient than the Hotelling's T^2 chart for detecting shifts of all sizes and in all the directions in the process location.
- For monitoring a process operating under a bivariate double exponential distribution with correlation $\rho = 0$, we observe that, the proposed control chart is more efficient than the Hotelling's T^2 chart for detecting shift of small sizes and in all the directions in the process location. When the correlation is $\rho = 0.6$, for detecting all shifts in all directions, the proposed control chart is more efficient.

Conclusion

In this study, a control chart based on Hodges sign test statistic is developed to monitor location of bivariate process. The performance of proposed chart is compared with parametric chart under bivariate normal and bivariate double exponential distributions. Simulation study indicates that the proposed nonparametric control chart is more efficient than the Hotelling's T^2 chart for detecting shifts in bivariate process location when the underlying process distribution is heavy tailed.

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